Self Generating Runs

5304 [1965, 674]. *Proposed by William Kolakoski, Carnegie Institute of Technology*

Describe a simple rule for constructing the sequence

$$1221121221221121122121121221121121221221 \cdots .$$

What is the nth term? Is the sequence periodic?

Solution by Necdet Üçoluk, Clarkson College of Technology. Let $\{x_n\}$ be a sequence not eventually constant. One can consider a sequence of blocks over $\{x_n\}$ by grouping all consecutive equal numbers in the same blocks. The lengths of these blocks will give a sequence $\{\hat{x}_n\}$ associated with $\{x_n\}$. The given sequence $\{a_n\}$ can be defined as follows: $a_1=1$, $a_n=1$ or 2, and $\{a_n\}=\{\hat{a}_n\}$, for all $n=1, 2, \cdots$. So $\{a_n\}$ is constructed uniquely. Indeed, $a_1=1$, so $\hat{a}_1=1$, hence the first block of $\{a_n\}$ is of length only 1, therefore $a_2\neq 1$, so $a_2=2$. Since $\hat{a}_2=a_2=2$, the second block in $\{a_n\}$ will consist of two 2's, so a_3 and then $\hat{a}_3$ equals 2. Since the third block in $\{a_n\}$ must start with 1, it will consist of two 1's, because $\hat{a}_3=2$. Therefore $a_4=a_5=1$. Continuing in this fashion, $\{a_n\}$ is constructed recursively. If $\{s_n\}$ is the sequence of partial sums of $\sum \hat{a}_n$, then

$a_n=\frac{1}{2}(3+(-1)^m)$ with $s_{m-1}<n\leqq s_m$, since consecutive blocks contain different numbers, odd-numbered blocks contain 1's while the others consist of 2's.

$\{a_n\}$ cannot be periodic. If $\{a_n\}$ were a periodic sequence, say after $n=n_0$, with the minimum period N, then $\{\hat{a}_n\}$ would be periodic after n_0 with the period N. The periodicity of $\{\hat{a}_n\}$ would induce another period $N_1>N$ for $\{a_n\}$ after a certain index n_1 (actually $n_0<n_1<2n_0$). But $N<N_1<2N$, since $\{a_n\}$ can have only blocks of length one or two, therefore a segment of $\{\hat{a}_n\}$ having N elements produces a segment in $\{a_n\}$ of length more than N but less than $2N$. This contradicts the fact that N is the minimum period for $\{a_n\}$. But N_1 must also be a multiple of N, and the non-periodicity of $\{a_n\}$ now follows from this contradiction.

Also solved by Walter Bluger, H. Brandt Corstius (Netherlands), Paul Cull, Jack Dix, R. F. Jackson, Norman Miller, Julius Nadas, C. E. Olson, C. B. A. Peck, J. R. Purdy, Donald Quiring, and Judith Richman.

Nadas generalized the given problem by considering sequences as above but using r integers.

Miller observed that the first 42 digits in the given sequence of 1's and 2's is obtained when each vowel is replaced by 1, each consonant by 2 in the following:

Indian and Ethiopian, Syrian, Israeli, Arab, Persian.